GRAPHING LINES

- Want some practice with preliminary concepts first?
  Introduction to Equations and Inequalities in Two Variables
  Introduction to the Slope of a Line
  Practice with Slope

(more mathematical cats)

In this section, we firm up the relationship between a line in the coordinate plane and its description as an equation in two variables.

In the process, some general strategies for graphing a line are discussed.

DEFINITION *linear equation in two variables*

A ***linear equation in two variables*** (x and y) is an equation of the form:

$$ax + by + c = 0$$

In this equation, a , b , and c are real numbers.
The numbers a and b cannot both equal zero.

Every linear equation in two variables graphs as a line in the coordinate plane.
Every line in the coordinate plane has a description as a linear equation in two variables.

The equation $ax + by + c = 0$ is often called the ***standard form*** or ***general form*** of a line.

IMPORTANT THINGS TO KNOW ABOUT LINEAR EQUATIONS IN TWO VARIABLES

- WHY AN EQUATION?
  The sentence $ax + by + c = 0$ is an ***equation*** because of the $=$ sign.
- WHY LINEAR?
  The sentence $ax + by + c = 0$ is ***linear*** because the variables (x and y) appear as simply as possible.
  Only to the first power. No variables in denominators. No variables under square roots. And so on.
  The word ‘linear’ makes perfect sense here, since every equation in two variables graphs as a line.

- HIGHER-LEVEL MEANING OF LINEAR:
  You should be aware that the word ‘linear’ is also used in higher-level situations.
  For fun, jump up to wolframalpha.com and type in this linear equation in ***three*** variables:
  `2x + 3y + 4z + 5 = 0`
  (Just cut-and-paste, if you want.)
  You'll see that it graphs as a plane in space!
- ‘AN EQUATION OF THE FORM’ ALLOWS FOR:
  When mathematicians say ***an equation of the form ...*** they really mean
  an equation that can be put in the form ...
  using the Addition and Multiplication properties of equality.

  Thus, for example, $y = 2x - 3$ is a linear equation in two variables; rewrite as: $2x - y - 3 = 0$
  Or, $-4y + 5x = 7$ is a linear equation in two variables; rewrite as: $5x - 4y - 7 = 0$
- ALTERNATE DEFINITION:
  Some people define a linear equation in two variables as an equation
  that can be written in the form $ax + by = c$ (with the same restrictions on a, b, and c).
  The two forms describe precisely the same set of equations.
  It's just a matter of preference. Use whichever you and your teacher prefer.
- WHAT HAPPENS IF BOTH a AND b ARE ZERO?
  If both a and b are zero, then both x and y disappear.
  You're left looking at an equation that's either always true ($0 = 0$) or always false (like $5 = 0$).
  What happens if we treat these as equations in two variables?
  The solution set of $0x + 0y + 0 = 0$ is the entire coordinate plane—all points satisfy the equation.
  The solution set of $0x + 0y + 5 = 0$ is empty—no points satisfy the equation.
  We definitely don't want these situations, which is why a and b can't both be zero.
- HORIZONTAL LINES:
  The number a can be zero, as long as b isn't zero.
  Then, you only ‘see’ the variable y.
  In this case, the equation can be rewritten to look like ‘ $y = \text{some number}$ ’, say, $y = k$.
  The solutions are all ordered pairs of the form (x, k), for all real numbers x, which is a horizontal line.
  You'll get more practice with this situation in Horizontal and Vertical Lines.
- VERTICAL LINES:
  The number b can be zero, as long as a isn't zero.
  Then, you only ‘see’ the variable x.
  In this case, the equation can be rewritten to look like ‘ $x = \text{some number}$ ’, say, $x = k$.
  The solutions are all ordered pairs of the form (k, y), for all real numbers y, which is a vertical line.
  You'll get more practice with this situation in Horizontal and Vertical Lines.

- THE INTERCEPT METHOD FOR GRAPHING A LINE:
  Once you know you're dealing with a line, you only need two points.
  Every non-horizontal, non-vertical line crosses the x-axis and y-axis at exactly one point.
  These two points are easy to get, because one of their coordinates is zero.

  An ***x-intercept*** is where a graph crosses the x-axis.
  Points on the x-axis have their y-value equal to zero.
  Thus, to find the x-intercept, set $y = 0$ and solve for x.

  A ***y-intercept*** is where a graph crosses the y-axis.
  Points on the y-axis have their x-value equal to zero.
  Thus, to find the y-intercept, set $x = 0$ and solve for y.

  Using these two special points to graph a line is often called the ***intercept method***.
- FINDING THE SLOPE OF THE LINE $ax + by + c = 0$:
  If $b \neq 0$, then you're dealing with a non-vertical line.
  You may need to know its slope;
  one good way to get it is to rewrite the equation in $y = mx + b$ form.
  Recall that in this form, the number m (the coefficient of the x term) gives the slope of the line.
  The form $y = mx + b$ is given a special name:

SLOPE-INTERCEPT FORM OF A LINE, $y = mx + b$

Every equation of the form $y = mx + b$ graphs as a non-vertical line.
The slope of the line is m (the coefficient of the x term).
The line crosses the y-axis at the point $(0, b)$.
Since the equation $y = mx + b$ so clearly displays the slope and y-intercept,
it is called ***slope-intercept form***.

IMPORTANT THINGS TO KNOW ABOUT SLOPE-INTERCEPT FORM

- CHECK THE y-INTERCEPT:
  Where does $y = mx + b$ cross the y-axis?
  Set $x = 0$, giving $y = m(0) + b = b$.
  Some people say: ‘ the y-intercept is b ’.
  Some people say: ‘ the y-intercept is $(0, b)$ ’.
  Both give unambiguous information about where the line crosses the y-axis.
  Use the terminology that you and/or your teacher prefers.
- GRAPHING A LINE FROM SLOPE-INTERCEPT FORM:
  Here's an easy way to graph $y = mx + b$:
  - Plot the point $(0, b)$. This is your y-intercept.
    You only need one more point to draw the line.
  - Rename the slope (as needed) as a fraction.
    For example, if the slope is 3, you might rename it as $\frac{3}{1}$.
  - Interpret the slope as $\frac{\text{rise}}{\text{run}}$;
    use this information to move from the y-intercept to a second point.
    For example, if the slope is $3 = \frac{3}{1}$, move up 3 and to the right 1.
  - Draw the line through your two points. Done!

EXAMPLES:

Question:
Consider the line $2x - 3y + 5 = 0$.
Write the equation in the form $y = mx + b$.
What is the slope of the line?
What is the y-intercept?
If you start at any point on the line, how could you move to get to another point?
Solution:
To put the equation in $y = mx + b$ form, solve for y:

$2x - 3y + 5 = 0$	(original equation)
$-3y + 5 = -2x$	(subtract $2x$ from both sides)
$-3y = -2x - 5$	(subtract 5 from both sides)
$y = \dfrac{-2x-5}{-3}$	(divide both sides by -3)
$y = \dfrac{2}{3}x + \dfrac{5}{3}$	(write in the most conventional way)

slope: $m = \dfrac{2}{3} = \dfrac{\text{rise}}{\text{run}}$

y-intercept: $b = \dfrac{5}{3}$

To get to a new point, you could move **up 2 and to the right 3**.
(There are, of course, other correct answers.)

Question:
Consider the line $2x - 3y + 5 = 0$.
What is the x-intercept? (Give the coordinates.)
What is the y-intercept? (Give the coordinates.)
Solution:
To find the x-intercept, set $y = 0$ and solve for x:

$2x - 3y + 5 = 0$	(original equation)
$2x - 3(0) + 5 = 0$	(set $y = 0$)
$2x = -5$	(subtract 5 from both sides)
$x = -\dfrac{5}{2}$	(divide both sides by 2)

The x-intercept is $(-\frac{5}{2}, 0)$.

To find the y-intercept, set $x = 0$ and solve for y:

$2x - 3y + 5 = 0$	(original equation)
$2(0) - 3y + 5 = 0$	(set $x = 0$)
$-3y = -5$	(subtract 5 from both sides)
$y = \dfrac{5}{3}$	(divide both sides by -3)

The y-intercept is $(0, \frac{5}{3})$.